

Notes on Derivation of 'Generalized Gravitational Entropy'

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Abstract

An alternative derivation of generalized gravitational entropy associated to co-dimension 2 'entangling' hypersurfaces is given. The approach is similar to the Jacobson-Myers 'Hamiltonian' method and it does not require computations on manifolds with conical singularities. It is demonstrated that the entangling surfaces should be extrema of the entropy functional. When our approach is applied to Lovelock theories of gravity the generalized entropy formula coincides with results derived by other methods.

1 Introduction

There are a mounting number of arguments that the Bekenstein-Hawking entropy can be applied not only in case of black hole horizons but to arbitrary co-dimension 2 surfaces in flat and curved spacetimes. First arguments that this can be done in a consistent way have been presented in the work of the present author [1],[2]. If \mathcal{B} is a minimal hypersurface in a constant time slice Σ of a stationary spacetime \mathcal{M} which is a solution to the Einstein theory one can associate to this surface an entropy [2]

$$S(\mathcal{B}) = \frac{A(\mathcal{B})}{4G} \quad , \quad (1.1)$$

where $A(\mathcal{B})$ is the area of \mathcal{B} . Equation (1.1) has been inspired by the holographic formula [3] for computing entanglement entropy in conformal theories with gravity duals. $S(\mathcal{B})$ can be interpreted as an entanglement entropy in quantum gravity [2]. A similar concept of spacetime entanglement was discussed in a number of publications, see e.g. [4], [5].

Recently formula (1.1) has been also proposed by Lewkowycz and Maldacena [6] as a 'generalized gravitational entropy'. The authors of [6] considered a general setup when \mathcal{M} is an arbitrary (not necessarily stationary) solution to the Einstein gravity. It was assumed that boundary $\partial\mathcal{M}$ of \mathcal{M} has non-contractable circles S^1 which are contractable inside \mathcal{M} on \mathcal{B} . When \mathcal{B} is minimal in \mathcal{M} equation (1.1) yields an entropy associated to a density matrix specified by the given boundary conditions. It was also argued that the above construction has an entanglement interpretation.

The Maldacena-Lewkowycz proposal and its extensions to higher derivative gravities attracted a considerable interest [7]-[12]. The main difficulty here was related to a careful treatment of conical singularities in gravity actions [13]. The singularities appeared in [6] at some steps of computations.

The aim of the present work is to derive the generalized gravitational entropy without any use of conical singularities. Our approach is similar to the Jacobson-Myers 'Hamiltonian' method [14] in a sense that the entropy appears from a boundary term in the action when one isolates a small domain around the 'entangling' surface \mathcal{B} . We prove the extremality of the entropy functional on the entangling surface and test our approach in Lovelock theories of gravity.

After necessary definitions in Sec. 2 the suggested method is introduced in Sec. 3. Applications to higher derivative gravities are considered in Sec. 4 followed by a brief discussion in Sec. 5.

2 Definitions

Entanglement entropy in a quantum gravity, as suggested in [2], is specified by the boundary conditions, which imply a holographic nature of the theory. One starts with a class of manifolds \mathcal{M} with the boundary condition $\partial\mathcal{M} = \mathcal{T}$, where \mathcal{T} is a $d-1$ dimensional manifold. The entanglement entropy of [1],[2] and the generalized gravitational entropy of [6] can be defined in terms of an 'entanglement' partition function $\mathcal{Z}[\mathcal{T}_n]$, where $n = 1, 2, \dots$, and \mathcal{T}_n are boundary manifolds constructed from n copies of \mathcal{T} . Construction of \mathcal{T}_n is similar to a construction of 'replicated' manifolds in a QFT to represent quantities like $\text{Tr } \rho^n$, where ρ is a reduced density matrix obtained by tracing over unobservable states.

The entanglement partition function $\mathcal{Z}[\mathcal{T}_n]$ is defined by quantum gravity theory, where bulk geometries \mathcal{M}_n have the boundary $\partial\mathcal{M}_n = \mathcal{T}_n$. One can represent $\mathcal{Z}[\mathcal{T}_n]$ in terms of some integral over 'histories' with above boundary conditions and integration measure defined by some low-energy action $I[\mathcal{M}_n]$. In a semiclassical approximation $\ln \mathcal{Z}[\mathcal{T}_n] \simeq -I[\bar{\mathcal{M}}_n]$, where $\bar{\mathcal{M}}_n$ realizes a minimum of the action for given boundary conditions, and the entropy can be defined as [2]

$$S = \lim_{n \rightarrow 1} (n \partial_n - 1) I[\bar{\mathcal{M}}_n] \quad . \quad (2.1)$$

One first finds the action for integer n , assumes that n can be replaced with a continuous parameter, and then goes to $n = 1$. This is a common trick used in statistical physics and known as a replica method.

The Maldacena-Lewkowycz approach is to look for $\bar{\mathcal{M}}_n$ as *regular* (except some 'harmless' singularities) solutions to the corresponding low-energy gravity equations with the condition $\partial\bar{\mathcal{M}}_n = \mathcal{T}_n$. It is assumed that $\bar{\mathcal{M}} = \bar{\mathcal{M}}_1$ is one of solutions for standard boundary conditions $\partial\bar{\mathcal{M}} = \mathcal{T}$. In the alternative approach [2] $\bar{\mathcal{M}}_n$ are allowed to have conical singularities. The idea of [2] is that gravity actions may have minima on such backgrounds if \mathcal{T}_n have conical singularities.

For the purposes of the present paper we follow [6]. Here the boundary manifolds \mathcal{T} are required to have non-contractable circles S^1 . One can introduce a coordinate τ along the circles with the period 2π . The boundary manifold \mathcal{T}_n for the partition function in the replica method is glued *smoothly* from n copies of \mathcal{T}_n such that τ has the period $2\pi n$. It is required that \mathcal{T} and \mathcal{T}_n are boundaries of manifolds where S^1 can be contracted in the bulk. A simple example is the case of a black hole instanton, where \mathcal{M} is a solid hypertorus for $\partial\mathcal{M} = S^1 \times S^{d-1}$.

In the rest of the paper we use the following notations: $R_{\mu\nu\lambda\rho}$ is the Riemann tensor of a d dimensional manifold \mathcal{M} , \mathcal{M} has the Euclidean signature. The Greek indexes run from 1 to d . $\partial\mathcal{M}$ is (either external or internal) boundary of \mathcal{M} , K_b^a is the extrinsic curvature tensor of $\partial\mathcal{M}$. The Latin indexes a, b, c, d run from 1 to $d - 1$. A tensor R_{abcd} on $\partial\mathcal{M}$ is a projection of the Riemann tensor of \mathcal{M} on a space tangent to $\partial\mathcal{M}$. \mathcal{B} is an 'entangling' co-dimension 2 hypersurface in \mathcal{M} . We use a unit complex vector constructed from two normal vectors to \mathcal{B} and define the corresponding complex extrinsic curvature k_j^i . The Riemann tensor defined by the metric of \mathcal{B} is denoted as \hat{R}_{ijkl} . The Latin indexes i, j, k, l run from 1 to $d - 2$. The operation $[\mu_1, \dots, \mu_p]$ denotes totally antisymmetric combination of p indexes (accompanied by the factor $1/p!$).

3 A novel derivation of the generalized entropy

To present the method we start with the Einstein gravity. Let $\bar{\mathcal{M}}_n$ be a solution to gravity equations for corresponding boundary conditions \mathcal{T}_n . By following [6] we assume that a Z_n symmetry of boundary conditions \mathcal{T}_n (permutations of replicas) is extended to the bulk. Let \mathcal{B}_n be a surface of fixed points in $\bar{\mathcal{M}}_n$ (points which do not move under the Z_n symmetry). Maldacena and Lewkowycz [6] interpret \mathcal{B}_n as a world-sheet of a cosmic string (brane) and derive conditions on \mathcal{B}_n from a regularity condition on the geometry around a cosmic string. We consider sets of solutions $\bar{\mathcal{M}}_n$ and corresponding surfaces \mathcal{B}_n but do not write the index n explicitly, for a while.

A 'cosmic string' action on \mathcal{B} can be inferred immediately from the gravity action on $\bar{\mathcal{M}}$. The idea is the following. Consider a small neighbourhood \mathcal{N}_ϵ around \mathcal{B} , where the metric, according to [6], behaves as

$$ds^2 \simeq r^2 d\tau^2 + n^2 dr^2 + \left(\gamma_{ij}(v) + 2r^n c^{1-n} (\cos \tau k_{ij}^{(1)}(v) + \sin \tau k_{ij}^{(2)}(v)) \right) dv^i dv^j . \quad (3.1)$$

Here $0 < \tau \leq 2\pi n$, $0 < r \leq \epsilon$, c is a dimensional constant, $\gamma_{ij}(v)$ is a metric on \mathcal{B} , and $k_{ij}^{(p)}(v)$ are two extrinsic curvatures of \mathcal{B} . Metric (3.1) is invariant under shifts $\tau \rightarrow \tau + 2\pi$. This property ensures the Z_n symmetry. The given metric does not have conical singularities in the (r, τ) part, and the geometry is regular for all natural n . For arbitrary n (for example, n slightly larger than 1) (3.1) is not regular at $r = 0$ due to terms with extrinsic curvatures. Components of the Ricci tensor have power-law divergences if $1 < n < 2$. In this work we adopt the point of view of [6] that the singularity due to the extrinsic curvature terms is 'harmless' in a sense it disappears in components of the Einstein tensor if \mathcal{B} obeys certain conditions. For the Einstein gravity this condition is that \mathcal{B} is minimal (extrinsic curvatures $k_{ij}^{(p)}$ have vanishing traces).

In coordinates (3.1) the boundary of the neighbourhood is chosen to be located at $r = \epsilon$. The gravity action on $\bar{\mathcal{M}}$ is decomposed on the action on \mathcal{N}_ϵ and the action on $\bar{\mathcal{M}}/\mathcal{N}_\epsilon$. It is assumed that a necessary boundary term with the extrinsic curvature on the boundary of \mathcal{N}_ϵ is included in the actions to have a well-posed variational problem. In the limit $\epsilon \rightarrow 0$ the action on \mathcal{N}_ϵ can be interpreted as a 'cosmic string' action I_{str} ,

$$I_{\text{str}}[\mathcal{B}] = \lim_{\epsilon \rightarrow 0} I[\mathcal{N}_\epsilon] = -A(\mathcal{B})/(4G) , \quad (3.2)$$

$$I[\mathcal{N}_\epsilon] = -\frac{1}{16\pi G} \int_{\mathcal{N}_\epsilon} \sqrt{g} d^d x R - \frac{1}{8\pi G} \int_{\partial \mathcal{N}_\epsilon} \sqrt{h} d^{d-1} y K . \quad (3.3)$$

To get (3.2) from (3.3) one should take into account that the extrinsic curvature tensor K_ν^μ of \mathcal{N}_ϵ has a singular component $K_\tau^\tau = 1/(n\epsilon)$. This singularity is compensated by the factor ϵ in the integration measure. The bulk part of $I[\mathcal{N}_\epsilon]$ vanishes in this limit. The 'cosmic string' has the negative tension $-1/(4G)$. Thus, the use of this terminology is only for an analogy, not for drawing any physical consequences. We also note that our definition of the 'string' differs from how it was originally introduced in [6],[10]. Our 'string' has a finite tension when $n \rightarrow 1$.

In the limit $\epsilon \rightarrow 0$ one can write

$$I[\bar{\mathcal{M}}] = I[\bar{\mathcal{M}}^c] + I_{\text{str}}[\mathcal{B}] , \quad (3.4)$$

where $I[\bar{\mathcal{M}}^c]$ is an action on a manifold $\bar{\mathcal{M}}^c = \bar{\mathcal{M}}/\mathcal{B}$, where \mathcal{B} is removed. Variation of (3.4) over the metric yields the Einstein equations outside \mathcal{B} . These are the vacuum equations if the matter is absent.

Variation of the 'string action' is easy to understand at a small but finite ϵ (at a finite string thickness). There are non-trivial variations on the boundary \mathcal{N}_ϵ due to the boundary terms in the gravity action on $\bar{\mathcal{M}}/\mathcal{N}_\epsilon$ and in the 'string' domain \mathcal{N}_ϵ . This yields equations

$$(K^{\mu\nu} - h^{\mu\nu} K)_+ = -(K^{\mu\nu} - h^{\mu\nu} K)_- . \quad (3.5)$$

Here $(K_+)_\nu^\mu$ and $(K_-)_\nu^\mu = K_\nu^\mu$ are the extrinsic curvatures of \mathcal{N}_ϵ in $\bar{\mathcal{M}}/\mathcal{N}_\epsilon$ and \mathcal{N}_ϵ , respectively. The left hand side comes out from the 'gravity part' and the right hand side

from the 'string'. The r.h.s. of (3.5) can be interpreted as a 'stress-energy tensor' of the 'string'. Equations (3.5) are identities since the division on the gravity and 'string' parts is artificial.

From now on the index n is restored. Before applying formula (2.1) we discuss variation of $I[\bar{\mathcal{M}}_n]$ over n . We use the same arguments as in [6] and consider $I[\bar{\mathcal{M}}_n]$ as some integrals at continuous n .

Let us start with decomposition (3.4). For $I[\bar{\mathcal{M}}_n^c]$ extrapolation to continuous n does not pose a problem since a small domain near \mathcal{B}_n is excluded. Variation over n can be written as

$$\partial_n I[\bar{\mathcal{M}}_n^c] = \partial_n^{\text{int}} I[\bar{\mathcal{M}}_n^c] + \partial_n^{\text{bulk}} I[\bar{\mathcal{M}}_n^c] + \partial_n^{\text{boun}} I[\bar{\mathcal{M}}_n^c] \quad . \quad (3.6)$$

The operation ∂_n^{int} means a change of the upper limit in the integrals in τ , when the integrand itself is fixed. This is equivalent to changing the number of replicas or the periodicity of τ . Variations ∂_n^{bulk} , ∂_n^{boun} take into account, respectively, change of metrics in the bulk and on the boundaries of \mathcal{M}_n^c (when the period of τ is fixed.) Variation of the string action can be written as

$$\partial_n I_{\text{str}}[\mathcal{B}_n] = \partial_n^{\text{metr}} I_{\text{str}}[\mathcal{B}_n] + \partial_n^{\text{pos}} I_{\text{str}}[\mathcal{B}_n] \quad , \quad (3.7)$$

where ∂_n^{metr} corresponds to the variation of the metric of \mathcal{B}_n , while ∂_n^{pos} takes into account change in the position of \mathcal{B}_n under fixed metric. If \mathcal{B}_n is a minimal surface the change of the position does not change the string action in the leading order.

We need variations at $n = 1$. Since the Z_n symmetry is implied

$$\lim_{n \rightarrow 1} \partial_n^{\text{int}} I[\bar{\mathcal{M}}_n^c] = I[\bar{\mathcal{M}}^c] \quad , \quad (3.8)$$

where $\bar{\mathcal{M}}^c = \bar{\mathcal{M}}_1^c$. Equation (3.8) is easy to understand when the metric does not depend on τ . One also has

$$\lim_{n \rightarrow 1} \partial_n^{\text{bulk}} I[\bar{\mathcal{M}}_n^c] = 0 \quad . \quad (3.9)$$

The action has an extremum on $\bar{\mathcal{M}}_n^c$.

Since the metric on the external boundary is fixed one should care about variations on the internal boundary of $\bar{\mathcal{M}}_n^c$. The latter are compensated by the variations of the string action,

$$\partial_n^{\text{metr}} I_{\text{str}}[\mathcal{B}_n] + \partial_n^{\text{boun}} I[\bar{\mathcal{M}}_n^c] = 0 \quad . \quad (3.10)$$

Eq. (3.10) is ensured by gravity equations (3.5). There is a subtle point here. Variations of the parameter n in the metric under fixed periodicity of τ result in conical singularities in (3.1), see [6]. Thus, (3.10) is satisfied up to terms $O(n - 1)$. There is no real cosmic string to support the singularity. Therefore, (3.10) holds only in the limit $n \rightarrow 1$, which is enough for our purposes.

By taking into account equations (3.7), (3.8), (3.9), (3.10) one finds

$$\lim_{n \rightarrow 1} \partial_n I[\bar{\mathcal{M}}_n] = I[\bar{\mathcal{M}}] + \lim_{n \rightarrow 1} \partial_n^{\text{pos}} I_{\text{str}}[\mathcal{B}_n] \quad , \quad (3.11)$$

$$S = -I_{\text{str}}[\mathcal{B}] + \lim_{n \rightarrow 1} \partial_n^{\text{pos}} I_{\text{str}}[\mathcal{B}_n] \quad . \quad (3.12)$$

The Bekenstein-Hawking formula (1.1) follows from (3.12) if one uses (3.2) and assumes that \mathcal{B} is a minimal surface ($\partial_n^{\text{pos}} I_{\text{str}}[\mathcal{B}_n] = 0$).

4 Entropy formula in the Lovelock gravity

From Eqs. (3.2), (3.12) the generalized entropy can be written as

$$S = -I_{\text{str}}[\mathcal{B}] = -\lim_{\epsilon \rightarrow 0} I[\mathcal{N}_\epsilon] \quad , \quad (4.1)$$

and it is a pure boundary term. This equality does not require that the theory is of the Einstein form. It can be also applied to higher derivative gravities provided that: *a*) the action functional admits boundary terms which insure well-posed variational procedure (normal derivatives of the metric variations do not appear on the boundary); *b*) the theory admits solutions $\bar{\mathcal{M}}_n$ for the given boundary conditions $\partial\bar{\mathcal{M}}_n = \mathcal{T}_n$ with the required Z_n symmetry; *c*) \mathcal{B} is an extremum of $I_{\text{str}}[\mathcal{B}]$ (remember that this condition eliminates the last term in the r.h.s. of (3.12)); *d*) singularities of the solutions near fixed point surfaces \mathcal{B}_n are 'harmless' in a sense that divergences in the gravity equations (in the higher curvature analogue of the Einstein tensor) are eliminated by certain conditions on \mathcal{B}_n , and these conditions at $n = 1$ are equivalent to equations which follow from requirement (*c*).

An example of a higher derivative gravity, where (*a*) is satisfied, is the Lovelock theory

$$I_L[\mathcal{M}] = -\sum_m c_m \left(\int_{\mathcal{M}} \sqrt{g} d^d x L_m + \int_{\partial\mathcal{M}} \sqrt{h} d^{d-1} y B_m \right) \quad . \quad (4.2)$$

Here c_m are some coefficients, $c_1 > 0$, and

$$L_m = \frac{(2m)!}{2^m} R_{[\mu_1 \nu_1}^{\mu_1 \nu_1} R_{\mu_2 \nu_2}^{\mu_2 \nu_2} \dots R_{\mu_m \nu_m}^{\mu_m \nu_m]} \quad , \quad (4.3)$$

$$B_m = \frac{(2m)!}{2^{m-1}} \sum_{p=0}^{m-1} d_{m,p} K_{[a_1}^{a_1} K_{a_2}^{a_2} \dots K_{a_{2p+1}}^{a_{2p+1}} R_{b_1 c_1}^{b_1 c_1} R_{b_2 c_2}^{b_2 c_2} \dots R_{b_{m-p-1} c_{m-p-1}}^{b_{m-p-1} c_{m-p-1}}] \quad , \quad (4.4)$$

$$d_{m,p} = \frac{(m-1)! 2^{3p} p!}{(m-p-1)!(2p+1)!} \quad . \quad (4.5)$$

It is implied that $R_{\mu'\nu'}^{\mu\nu} = R^{\mu\nu}_{\mu'\nu'}$, $R_{a'b'}^{ab} = R^{ab}_{a'b'}$. Curvatures in the r.h.s. of (4.4) are taken on \mathcal{B} . We use the form of the boundary term (4.4) given in [16].

Consider the Lovelock action in a small domain \mathcal{N}_ϵ , where the metric behaves as in (3.1). As earlier, we place the boundary $\partial\mathcal{N}_\epsilon$ at $r = \epsilon$. The 'string action' in this theory is determined by the boundary terms on $\partial\mathcal{N}_\epsilon$.

We need to study boundary terms in (4.2) in the limit $\epsilon \rightarrow 0$. Since the only singular component of K_b^a is $K_\tau^\tau = 1/\epsilon$ one can easily see that $B_m \sim 1/\epsilon$ at $\epsilon \rightarrow 0$. The singular terms can be easily extracted from (4.4):

$$K_{[a_1}^{a_1} \dots K_{a_{2p+1}}^{a_{2p+1}} R_{b_1 c_1}^{b_1 c_1} \dots R_{b_{m-p-1} c_{m-p-1}}^{b_{m-p-1} c_{m-p-1}}] \simeq \frac{2p+1}{2m-1} K_\tau^\tau K_{[i_1}^{i_1} \dots K_{i_{2p}}^{i_{2p}} R_{j_1 k_1}^{j_1 k_1} \dots R_{j_{m-p-1} k_{m-p-1}}^{j_{m-p-1} k_{m-p-1}}] \quad . \quad (4.6)$$

The factor $2p+1$ in the r.h.s. of (4.6) appears since a pair of upper and lower τ indexes take $2p+1$ positions, $2m-1$ in the denominator results from the normalization factor in the operator [...]. The indexes i, j, k enumerate components of the curvature tensors in the directions tangent to \mathcal{B} .

It is convenient to introduce complex extrinsic curvatures of \mathcal{B}

$$k_{ij} = \frac{1}{2}(k_{ij}^{(1)} - i k_{ij}^{(2)}) \quad , \quad \bar{k}_{ij} = k_{ij}^* \quad (4.7)$$

and use the relation, which follows from (3.1) at $n = 1$,

$$K_{ij} = e^{i\tau} k_{ij} + e^{-i\tau} \bar{k}_{ij} \quad (4.8)$$

(we assume $n = 1$ for the r.h.s. of (4.1)). Integration over the τ coordinate can be easily done,

$$\begin{aligned} \int_0^{2\pi} d\tau K_{[a_1}^{[a_1} \dots K_{a_{2p+1}}^{a_{2p+1}} R_{b_1 c_1}^{b_1 c_1} \dots R_{b_{m-p-1} c_{m-p-1}}^{b_{m-p-1} c_{m-p-1}}] \simeq \\ \frac{2\pi}{\epsilon} \frac{(2p+1)}{(2m-1)} \frac{(2p)!}{p!} k_{[i_1}^{[i_1} \dots k_{i_p}^{i_p} \bar{k}_{i_{p+1}}^{i_{p+1}} \dots \bar{k}_{i_{2p}}^{i_{2p}} R_{j_1 k_1}^{j_1 k_1} \dots R_{j_{m-p-1} k_{m-p-1}}^{j_{m-p-1} k_{m-p-1}}] \quad . \end{aligned} \quad (4.9)$$

The factor $(2p)!/p!$ in the r.h.s. of (4.9) counts the number of ways when p k -curvatures (or \bar{k} -curvatures) appear from $2p$ K -curvatures.

When (4.9) is used in the boundary term (see (4.4)) one comes to the action

$$\lim_{\epsilon \rightarrow 0} I_L[\mathcal{N}_\epsilon] = -4\pi \sum_m m c_m \hat{I}_m[\mathcal{B}] \quad , \quad (4.10)$$

$$\hat{I}_m[\mathcal{B}] = \int_{\mathcal{B}} \hat{L}_{m-1} \quad , \quad (4.11)$$

$$\begin{aligned} \hat{L}_{m-1} = \frac{(2(m-1))!}{2^{m-1}} \sum_{p=0}^{m-1} \frac{2^{3p}(m-1)!}{p!(m-p-1)!} \\ k_{[i_1}^{[i_1} \dots k_{i_p}^{i_p} \bar{k}_{i_{p+1}}^{i_{p+1}} \dots \bar{k}_{i_{2p}}^{i_{2p}} R_{j_1 k_1}^{j_1 k_1} \dots R_{j_{m-p-1} k_{m-p-1}}^{j_{m-p-1} k_{m-p-1}}] \quad . \end{aligned} \quad (4.12)$$

One can now see that the last equation (4.12) is of the Lovelock form on \mathcal{B} ,

$$\hat{L}_{m-1} = \frac{(2(m-1))!}{2^{m-1}} \hat{R}_{[i_1 j_1}^{[i_1 j_1} \hat{R}_{i_2 j_2}^{i_2 j_2} \dots \hat{R}_{i_{m-1} j_{m-1}}^{i_{m-1} j_{m-1}}] \quad . \quad (4.13)$$

Eqs. (4.11), (4.12) follow from (4.13) if one uses in (4.13) the Gauss-Codazzi equations on \mathcal{B}

$$\hat{R}_{i_1 j_2}^{j_1 j_2} = R_{i_1 i_2}^{j_1 j_2} + 2(k_{i_1}^{j_1} \bar{k}_{i_2}^{j_2} + \bar{k}_{i_1}^{j_1} k_{i_2}^{j_2} - k_{i_2}^{j_1} \bar{k}_{i_1}^{j_2} - \bar{k}_{i_2}^{j_1} k_{i_1}^{j_2}) \quad . \quad (4.14)$$

Factor $(m-1)!/(p!(m-p-1)!)$ yields a number of ways to pick up p $k\bar{k}$ -pairs. Multiplier 2^{3p} takes into account factor 2 in the r.h.s. of (4.14) and the fact that each Riemann curvature in (4.14) produces 4 $k\bar{k}$ -pairs.

We come to the following formula of the generalized entropy associated to the surface \mathcal{B} :

$$S = 4\pi \sum_m m c_m \hat{I}_m[\mathcal{B}] \quad . \quad (4.15)$$

In a context of the holographic entanglement entropy (4.15) has been suggested in [15]. In case of the Gauss-Bonnet gravity this entropy formula has been obtained by different methods: in [13] by using regularized conical singularity method and in [7],[8] from the requirement of regularity of the geometry around the 'cosmic string'. For Lovelock gravities (4.15) was also derived in [10].

There are arguments [10] that the Lovelock gravity satisfies condition (d) if \mathcal{B} is an extremum of (4.15). A careful study of this property in case of the Gauss-Bonnet gravity can be found in [12].

5 Discussion

We presented a sketch of arguments which may support the Maldacena-Lewkowycz proposal [6] when the low-energy gravity action has higher derivatives. We have not yet emphasized but implied that this construction should be also applicable to holographic entanglement entropy. In this case \mathcal{B} is a holographic entangling surface and the background manifold \mathcal{M} is a solution to an AdS gravity.

Our arguments (and, perhaps, other derivations of the generalized gravitational entropy) cannot be considered as a sort of a mathematical proof. One should demonstrate that gravity solutions for given boundary conditions for each value of the replica parameter n do exist and obey condition (d) formulated in sec. 4. If this is the case the generalized entropy can be derived as a limiting value of a boundary term in the action. The derivation is self-consistent if the entangling surface is an extremum of the entropy functional.

One should mention that (d) may not be respected in arbitrary higher derivative gravities [12].

In contrast to [6] the approach of [2] operates with singular geometries. By the construction, the bulk manifolds \mathcal{M}_n in [2] are replicas of \mathcal{M}_1 with conical singularities at \mathcal{B} . The two ways, [2] and [6], lead to the same entanglement entropy but yield different results for the Renyi entropies. It may happen that the two approaches compliment each other and the choice between them is determined by studying for which background the gravity action has a least value.

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